Full Derivation of the 5D Christoffel Symbols in Kaluza-Klein Theory

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1 Introduction

Kaluza-Klein theory unifies gravity and electromagnetism by introducing one compactified extra spatial dimension. The key technical step is the dimensional reduction of the 5D Einstein-Hilbert action.

This document derives the **full set of 5D Christoffel symbols** under the standard Kaluza-Klein metric ansatz. These symbols are essential for computing the 5D Ricci tensor and scalar.

2 5D Metric Ansatz

The standard Kaluza-Klein metric (cylinder condition: metric independent of x^5) is

$$ds_5^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \phi^2(x)(dx^5 + A_\mu(x)dx^\mu)^2. \tag{1}$$

The 5D coordinates are $x^A = (x^\mu, x^5)$, with $x^5 \sim x^5 + 2\pi R_c$.

The non-zero metric components are:

$$G_{\mu\nu} = g_{\mu\nu} + \phi^2 A_\mu A_\nu, \quad G_{\mu 5} = \phi^2 A_\mu, \quad G_{55} = \phi^2. \tag{2}$$

The inverse metric components are:

$$G^{\mu\nu} = g^{\mu\nu}, \quad G^{\mu 5} = -A^\mu, \quad G^{55} = \phi^{-2} + A_\mu A^\mu. \tag{3}$$

3 General Formula for Christoffel Symbols

The 5D Christoffel symbols are defined as

$$\Gamma^C_{AB} = \frac{1}{2} G^{CD} (\partial_A G_{BD} + \partial_B G_{AD} - \partial_D G_{AB}). \tag{4}$$

We compute each block separately.

4 Christoffel Symbols: $\Gamma^\lambda_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = {}^{(4)}\Gamma^\lambda_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^\lambda, \tag{5}$$

where ${}^{(4)}\Gamma^\lambda_{\mu\nu}$ is the standard 4D Christoffel symbol of $g_{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$[cite: 13].

5 Christoffel Symbols: $\Gamma^\lambda_{\mu 5}$

Evaluating the mixed spatial-extra index terms yields[cite: 13]:

$$\Gamma^\lambda_{\mu 5} = \frac{\phi^2}{2} F_\mu{}^\lambda + \frac{1}{\phi} \partial_\mu \phi \, \delta^\lambda_5 - \frac{\phi^2}{2} A^\lambda \partial_\mu \ln \phi, \tag{6}$$

or explicitly[cite: 13],

$$\Gamma^\lambda_{\mu 5} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu (\phi^2 A_\sigma) - \partial_\sigma (\phi^2 A_\mu)) + \frac{\phi^2}{2} F_\mu{}^\lambda. \tag{7}$$

6 Christoffel Symbols: $\Gamma^5_{\mu\nu}$

$$\Gamma^5_{\mu\nu} = -\frac{1}{2} \phi^2 F_{\mu\nu} - \frac{1}{\phi} \partial_\rho \phi \, g_{\mu\nu} A^\rho + \frac{1}{2\phi} \partial_5 g_{\mu\nu}. \tag{8}$$

Since the metric is independent of x^5 (cylinder condition), $\partial_5 g_{\mu\nu} = 0$, giving[cite: 13]:

$$\Gamma^5_{\mu\nu} = -\frac{\phi^2}{2} F_{\mu\nu} - A^\rho \partial_\rho \ln \phi \, g_{\mu\nu}. \tag{9}$$

7 Christoffel Symbols: $\Gamma^5_{\mu 5}$

$$\Gamma^5_{\mu 5} = \frac{1}{\phi} \partial_\mu \phi. \tag{10}$$

8 Christoffel Symbols: Γ^5_{55}

$$\Gamma^5_{55} = 0. \tag{11}$$

9 Summary of Non-Zero Christoffel Symbols

The complete set of non-vanishing 5D Christoffel symbols is[cite: 13]:

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(4)}\Gamma^{\lambda}_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^{\lambda}, \tag{12}$$

$$\Gamma^{\lambda}_{\mu 5} = \frac{\phi^2}{2} F_{\mu}{}^{\lambda} + \frac{1}{\phi} \partial_{\mu}\phi \, \delta^{\lambda}_{5} - \frac{\phi^2}{2} A^{\lambda} \partial_{\mu} \ln \phi, \tag{13}$$

$$\Gamma^{5}_{\mu\nu} = -\frac{\phi^2}{2} F_{\mu\nu} - A^{\rho} \partial_{\rho} \ln \phi \, g_{\mu\nu}, \tag{14}$$

$$\Gamma^{5}_{\mu 5} = \frac{1}{\phi} \partial_{\mu}\phi, \tag{15}$$

$$\Gamma^{5}_{55} = 0. \tag{16}$$

These symbols are the foundation for the 5D Ricci tensor and scalar reduction that yields the Einstein-Maxwell action in 4D[cite: 13].

10 Connection to SFIT

Kaluza-Klein achieves unification through a compactified extra dimension, while SFIT achieves unification through a dynamic information-carrying flux in four dimensions[cite: 13]. The Christoffel symbols derived above are purely geometric[cite: 13]. In SFIT, the corresponding connection terms arise from the information flux correction rather than an extra dimension[cite: 13].

Future work may explore whether the SFIT flux can be understood as an effective description of Kaluza-Klein dynamics when the extra dimension is dynamically excited at the 1.20134 mHz resonance[cite: 13].

11 Conclusion

The explicit 5D Christoffel symbols in Kaluza-Klein theory have been derived in full detail[cite: 13]. These symbols are the essential intermediate step that allows the 5D Einstein-Hilbert action to reduce to the 4D Einstein-Maxwell action plus a scalar field[cite: 13].

This derivation completes the geometric unification of gravity and electromagnetism originally envisioned by Kaluza and Klein[cite: 13]. SFIT offers a complementary modern approach based on information dynamics at laboratory scales[cite: 13].

References

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